Nassau County Interscholastic Mathematics League

Contest #5 Answers must be integers from 0 to 999 inclusive. 2011 – 2012

No calculators are allowed.

Time: 10 minutes

1. Compute $\frac{6^6+6^6+6^6+6^6+6^6+6^6}{6\cdot6\cdot6\cdot6\cdot6}$.

2. Points A and B are in quadrant I in a Cartesian coordiate system with the origin as point O. The slopes of $\overline{OA}$ and $\overline{OB}$ are 1 and $\sqrt{7}$ respectively. If $OA = OB$, the slope of $\overline{AB}$ can be expressed in the form $p - \sqrt{q}$ where q is not divisible by the square of any prime. Compute $p + q$.

Time: 10 minutes

3. Compute the remainder when 19^5 is divided by 20.

4. Compute $\frac{1}{\log_5 240} + \frac{1}{\log_6 240} + \frac{1}{\log_8 240}$.

Time: 10 minutes

5. A sequence of positive integers is defined as follows: $a_1 = 37, a_2 = 71$, and $a_n = a_{n-1} + a_{n-2}$ for all $n \geq 3$. How many of the first 2012 terms are divisible by 3?

6. Find the sum of all values of the degree measures of x with $0° < x < 360°$ that satisfy $\sqrt{3}\sin x - \cos x = \sqrt{2}$.

Solutions for Contest #5

1. $\frac{6^6+6^6+6^6+6^6+6^6+6^6}{6\cdot 6\cdot 6\cdot 6\cdot 6\cdot 6} = \frac{6\cdot 6^6}{6^6} = 6.$

2. Let the coordinates of point A be (a, a) and the coordinates of point B be $(b, b\sqrt{7})$. Using the fact that $OA = OB$, we can use the square of the distance formula to get $a^2 + a^2 = b^2 + \left(b\sqrt{7}\right)^2$. So, $2a^2 = 8b^2$ or $a^2 = 4b^2$. Since the points are in quadrant I, $a = 2b$. The slope of $\overline{AB}$ is $\frac{b\sqrt{7}-a}{b-a} = \frac{b\sqrt{7}-2b}{b-2b} = \frac{\sqrt{7}-2}{-1} = 2 - \sqrt{7}$. Since B is in quadrant I, $b \neq 0$. The required sum is 9.

3. Use the binomial expansion formula to yield: $19^5 = (20-1)^5 = 20^5 - 5\cdot 20^4 + 10\cdot 20^3 - 10\cdot 20^3 + 5\cdot 20 - 1$. So, 19^5 is one less than a multiple of 20 and the requested remainder is 19.
<u>Alternate solution</u>: $19^5 \equiv (-1)^5 \equiv -1 \equiv 19 \pmod{20}$.

4. Use the change of base law:

$$\frac{1}{\log_5 240} + \frac{1}{\log_6 240} + \frac{1}{\log_8 240} = \frac{1}{\frac{\log 240}{\log 5}} + \frac{1}{\frac{\log 240}{\log 6}} + \frac{1}{\frac{\log 240}{\log 8}} = \frac{\log 5}{\log 240} + \frac{\log 6}{\log 240} + \frac{\log 8}{\log 240}$$
$$= \frac{\log 5 + \log 6 + \log 8}{\log 240} = \frac{\log(5\cdot 6\cdot 8)}{\log 240} = \frac{\log 240}{\log 240} = 1.$$

5. The remainders when the numbers are divided by 3 follow the same rule as the sequence, where the remainders are added modulo 3. The remainder sequence for the first eight terms is 1, 2, 0, 2, 2, 1, 0, 1. This sequence repeats infinitely. In 2012 terms there are 251 blocks of eight, each containing two zeroes (accounting for 2008 terms) and in the remaining four terms, there is one more zero. So there are $251\cdot 2 + 1 = 503$ terms with zero remainder when divided by three.

6. Notice the similarity of the given equation and the formula for $\sin(A-B) = \sin A \cos B - \sin \mathrm{B} \cos A$. Divide the given equation by two to get $\sin x \frac{\sqrt{3}}{2} - \frac{1}{2}\cos x = \frac{\sqrt{2}}{2}$. This is equivalent to $\sin x \cos 30° - \sin 30° \cos x = \frac{\sqrt{2}}{2}$. So, $\sin(x-30)° = \frac{\sqrt{2}}{2}$ and $x - 30 = 45$ or 135. So $x = 75$ or 165. The requested sum is 240.

Answers: 1. 6 2. 9 3. 19 4. 1 5. 503 6. 240