Nassau County Interscholastic Mathematics League

Contest #4 Answers must be integers from 0 to 999 inclusive. 2011 – 2012

Calculators are allowed.

Time: 10 minutes

1. Solve: $+2-(x+1)=x-\left(x-1-\left(x-2-\left(x-3-(x-4)\right)\right)\right)$.

2. A line reflection maps the point $A\ (-1,3)$ to point $A'(2,6)$ and maps point $B\ (-5,4)$ to point $B'(p,q)$. Compute $p+q$.

Time: 10 minutes

3. The perimeter of a parallelogram is 84 and the distances from a vertex of the parallelogram to two of its sides are 6 and 8. Compute the length of the shorter side.

4. Compute the number of ordered triples (a,b,c) of integers, where $a,b,$ and c are each between 1 and 10 inclusive, such that $2^a+3^b+4^c$ is a multiple of 3.

Time: 10 minutes

5. Compute the smallest integer which is a multiple of 18 and which is also a perfect cube.

6. The lengths of the bases of a trapezoid are 14 and 24 and the length of each leg is $\sqrt{386}$. If the length of the radius of the circumscribed circle is r, compute r^2.

Solutions for Contest #4

1. Simplify the equation to $1 = x - (x - 1 - (x - 2 - 1))$ and then to $1 = x - (x - 1 - (x - 3))$ and then to $1 = x - 2$. So $= 3$.

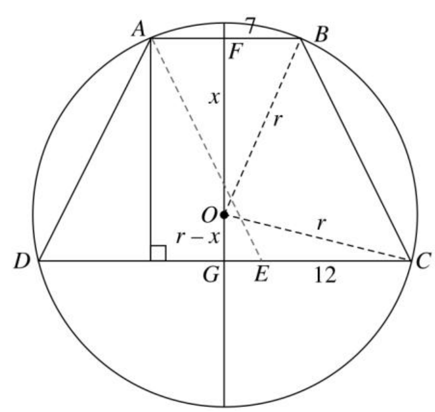
2. The line of reflection is the perpendicular bisector of both $\overline{AA'}$ and $\overline{BB'}$. To find its equation, let the midpoints of $\overline{AA'}$ and $\overline{BB'}$ be C and D respectively. The coordinates of C are $(0.5, 4.5)$ and the slope of $\overline{AA'}$ is 1. So the slope of the reflection line is -1. So, the equation of the reflection line is $y - 4.5 = -1(x - 0.5)$ or $y = 5 - x$. Since $\overline{AA'}$ and $\overline{BB'}$ are lines in the same plane that are perpendicular to the same line, they are parallel and their slopes are equal. So, the slope of $\overline{BB'}$ is $\frac{q-4}{p+5} = 1$. So $q = p + 9$. The coordinates of D are $\left(\frac{p-5}{2}, \frac{p+13}{2}\right)$. Substitute these coordinates into the equation of the reflection line to obtain $\frac{p+13}{2} = \frac{5-p}{2} + 5$ and $p = 1$. So, $B'(1,10)$ and the requested sum is 11.

3. The area of the parallelogram is $bh = 8x = 6(42 - x)$ and $x = 18$.

4. Use number congruence. First, $2^a + 3^b + 4^c \equiv (-1)^a + 0^b + 1^c \pmod{3}$. Since $2^a + 3^b + 4^c$ must be divisible by 3, $(-1)^a + 0^b + 1^c \equiv 0 \pmod{3}$. This is true only if a is odd. So there are five values of a and ten values for each of b and c that will give us the requested ordered triples. So, $5 \cdot 10 \cdot 10 = 500$.

5. $18 = 2 \cdot 3^2$. $(2 \cdot 3^2)(2^2 \cdot 3) = 2^3 \cdot 3^3 = 6^3 = 216$. Alternate solution: This problem can be done by trial and error on a calculator.

6. Call the trapezoid $ABCD$ with $AB = 14$. One way to find an altitude of the trapezoid is to draw a line through point A parallel to $\overline{BC}$ and intersecting $\overline{CD}$ at point E. Then $AE = BC = \sqrt{386}$ and $DE = DC - CE = 10$. In ΔADE the altitude from A, which is also an altitude of the trapezoid is $\sqrt{386 - 25} = 19$. The circum-center O of $ABCD$ lies on the perpendicular bisector of its bases. Let F and G be the points where the perpendicular bisector intersects $\overline{AB}$ and $\overline{CD}$ respectively. Let $OF = x$. In right ΔAOF, $7^2 + x^2 = AO^2 = r^2 = DO^2 = (19 - x)^2 + 12^2$. So, $x = 12$ and $r^2 = 7^2 + 12^2 = 193$. Without loss of generality, we placed the circum-center inside the trapezoid. Notice that the result is the same if we placed the circum-center outside the trapezoid.



Answers: 1. 3 2. 11 3. 18 4. 500 5. 216 6. 193