Nassau County Interscholastic Mathematics League

Contest #3 Answers must be integers from 0 to 999 inclusive. 2011 – 2012

No calculators are allowed.

Time: 10 minutes

1. The price of an item is increased by 20% and then decreased by 25%. The final price is \$2.34. Compute the number of cents in the original price of the item.

2. If $a = 2^{2011} + 2^{-2011}$ and $b = 2^{2011} - 2^{-2011}$, compute $a^2 - b^2$.

Time: 10 minutes

3. Compute the remainder when $772 \cdot 773 + 774$ is divided by 7.

4. The length of a side of an equilateral triangle is 6. The distances from a point inside the triangle to two of the sides are 1 and 3. Let d be the distance from the point to the third side. If $d = \sqrt{p} - q$, where p and q are positive integers, compute $p + q$.

Time: 10 minutes

5. Compute $\left(\sqrt[3]{17} - \sqrt[3]{13}\right)\left(\sqrt[3]{289} + \sqrt[3]{221} + \sqrt[3]{169}\right)$.

6. The value of x for which $\frac{2}{x} + 3x$ has its minimum positive value can be expressed as $\frac{\sqrt{p}}{q}$ where p and q are positive integers and p is not divisible by the square of any prime. Compute $p + q$.

Solutions for Contest #3

1. Let x be the number of cents in the original price. Then $(1.20)(0.75x) = 234$. So, $0.9x = 234$ and $x = 260$.

2. First note that $2^{2011} \cdot 2^{-2011} = 1$.
$a^2 - b^2 = (a+b)(a-b) = (2 \cdot 2^{2011})(2 \cdot 2^{-2011}) = 4 \cdot 1 = 4$.

3. Note that $772 = 7k + 2,\ 773 = 7k + 3$ and $774 = 7k + 4$. Also, note that $(7k+2)(7k+3) + 7k + 4 = 49k^2 + 42k + 10 = 7(7k^2 + 6k + 1) + 3$. So the requested remainder is 3. Alternate solution: Use number congruence: $772 \cdot 773 + 774 \equiv 2 \cdot 3 + 4 \equiv 10 \equiv 3 \pmod{7}$.

4. Draw a line segment from the point inside the given triangle to each of the three vertices to form three smaller triangles. The three distances from the point to each of the sides of the given triangle are lengths of altitudes of the smaller triangles. The area of the given equilateral triangle is $\frac{s^2\sqrt{3}}{4} = \frac{6^2\sqrt{3}}{4} = 9\sqrt{3}$. Since the sum of the areas of the small triangles is the area of the original equilateral triangle, $\frac{1}{2} \cdot 6 \cdot 1 + \frac{1}{2} \cdot 6 \cdot 3 + \frac{1}{2} \cdot 6 \cdot d = 9\sqrt{3}$. Then $d = 3\sqrt{3} - 4 = \sqrt{27} - 4$. So, $p + q = 31$. Alternate solution: Use the fact that the sum of the three distances from a point inside an equilateral triangle to its sides equals the length of the altitude of the triangle. Since the length of the side of the equilateral triangle is 6, the length of its altitude is $3\sqrt{3}$ and $1 + 3 + d = 3\sqrt{3}$, etc.

5. Let $x = \sqrt[3]{17}$ and $y = \sqrt[3]{13}$. Then $\left(\sqrt[3]{17} - \sqrt[3]{13}\right)\left(\sqrt[3]{289} + \sqrt[3]{221} + \sqrt[3]{169}\right) =$ $(x - y)(x^2 + xy + y^2) = x^3 - y^3 = 17 - 13 = 4$.

6. The arithmetic mean-geometric mean inequality states that the arithmetic mean of a set of positive numbers is greater than or equal to its geometric mean, with equality if and only if all the numbers in the set are equal. In this problem, $\frac{\frac{2}{x}+3x}{2} \geq \sqrt{\frac{2}{x} \cdot 3x} = \sqrt{6}$. The function has a minimum positive value (of $2\sqrt{6}$) when $\frac{2}{x} = 3x$ or when $x = \frac{\sqrt{6}}{3}$. The required sum is 9.

Answers: 1. 260 2. 4 3. 3 4. 31 5. 4 6. 9