Nassau County Interscholastic Mathematics League

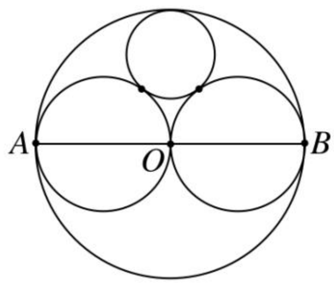
Contest #1 Answers must be integers from 0 to 999 inclusive. 2011 – 2012

No calculators are allowed.

Time: 10 minutes

1. Compute the units digit of $55^{55} + 65^{55}$.

2. A circle with center O has diameter $\overline{AB}$. Circles with diameters $\overline{AO}$ and $\overline{BO}$ are drawn. A fourth circle is externally tangent to the two smaller circles and internally tangent to the larger. Let p be the ratio of the area of the smallest of the four circles to the area of the largest of the four circles. Compute $1000p$ and round your answer to the nearest integer.



Time: 10 minutes

3. A hurricane flooded my basement. To pump out the water from my basement, I used a pump that could drain the basement in six hours. My neighbor gave me a pump that could drain my basement in half the time. Compute the number of hours that it would take both pumps working together to drain the basement.

4. In quadrilateral $ABCD, AB = BC = 10, CD = DA = 14$, and $BD = 12$. Let r be the radius of the inscribed circle of quadrilateral $ABCD$. Compute r^2.

Time: 10 minutes

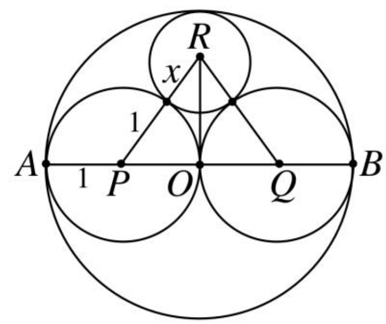
5. The difference of the roots of $x^2 + 4x + \frac{p}{q} = 0$ is 1, where p and q are relatively prime positive integers. Compute $p + q$.

6. Compute the sum of all values of the degree-measures of θ with $0° \leq \theta \leq 360°$ that satisfy $\cos\theta = -\frac{\csc\theta}{4}$.

Solutions for Contest #1

1. If the units digit of a number is 5, then if that number is raised to any exponent, then the units digit of the power is 5. The sum of two integers ending in five is an integer ending in 0. Alternatively, $55^{55} + 65^{65} \equiv 5 + 5 \equiv 0 \pmod{10}$.

2. Let P and Q be the centers of the two congruent circles. Let R be the center of the fourth circle and let x be its radius. Without loss of generality, let $AB = 4$. Then $PO = 1, PR = 1 + x,$ and $OR = 2 - x$. Then in ΔOPR, $1^2 + (2 - x)^2 = (1 + x)^2$. Therefore, $x = \frac{2}{3}$ and $1000p = 1000\frac{\pi\left(\frac{2}{3}\right)^2}{\pi(2)^2} = \frac{1000}{9} \approx 111.$



3. My neighbor's pump can drain the basement in three hours. In one hour, my pump can drain $\frac{1}{6}$ of the water in my basement and my neighbor's pump can drain $\frac{1}{3}$ of the water in my basement. Together, the pumps can drain $\frac{1}{6} + \frac{1}{3} = \frac{1}{2}$ of the water in my basement in one hour. So, it takes two hours to do the whole job with the pumps working together.

4. Let I be the center of the circle inscribed in quadrilateral $ABCD$. Since $\overleftrightarrow{BD}$ is a line of symmetry of the quadrilateral, point I must be on $\overline{BD}$. Let the points of tangency on $\overline{BC}$ and $\overline{DC}$ be X and Y respectively. Since $\Delta IXC \cong \Delta IYC$ (hy-leg), $\overline{CI}$ is an angle bisector of $\measuredangle BCD$. Using the angle-bisector theorem, $\frac{BI}{ID} = \frac{BC}{CD} = \frac{5}{7}$. Thus, the area of ΔBCI, $[BCI] = \frac{5}{12}[BCD]$. Use Heron's formula to find $[BCD] = 24\sqrt{6}$. Then $[BCI] = 10\sqrt{6}$.
   In ΔBCI, the altitude from I is an in-radius of $ABCD$. Call it r. Therefore, $[BCI] = 10\sqrt{6} = \frac{1}{2}10r$. So, $r^2 = 24$.

5. Call the roots $r_1$and r_2. $r_1 - r_2 = 1$ and $r_1 + r_2 = -4$. $\therefore r_1 = -\frac{3}{2}$ and $r_2 = -\frac{5}{2}$. $r_1 r_2 = \frac{p}{q} = \frac{15}{4}. \therefore p = 15$ and $q = 4$ and $p + q = 19$.

6. $\cos\theta = -\frac{\csc\theta}{4} = -\frac{1}{4\sin\theta}$. $2\sin\theta\cos\theta = -\frac{1}{2}$. $\sin 2\theta = -\frac{1}{2}$.
   $2\theta = 210° + 360°k$ or $330° + 360°k$, where k is an integer. To keep θ in the required interval $\theta = 105°, 285°, 165°$ or $345°$. The requested sum is 900.

Answers: 1. 0 2. 111 3. 2 4. 24 5. 19 6. 900