Time: 10 minutes

1. Find the greatest of 36 consecutive integers whose sum is 738.

2. How many sequences of at least two positive integers, not necessarily distinct, have a sum of 10? Notice that the sequence 3, 5, 1, 1 is not the same as the sequence 1, 1, 5, 3.

Time: 10 minutes

3. When a positive integer is divided by *d*, the remainder is 3. When the same division is done on a calculator, the digit to the right of the decimal point is also 3. Find the sum of all possible positive integer values of *d*.

4. If x and y are real numbers such that 3x + 4y = 12, and p/q is the minimum value of $x^2 + y^2$, where p and q are relatively prime positive integers, find $p + q$.

Time: 10 minutes

5. Two circles intersect each other at *A* and *B*. The radii of the circles are 10 and 17, and the distance between their centers is 21. Find *AB*.

6. If $\cos 20^\circ \cos 40^\circ \cos 80^\circ = p / q$, where p and q are relatively prime positive integers, find $p + q$.

Solutions for Contest #5

1. The median of the 36 integers is the same as the mean, which equals 738/36 = 20.5. Thus 18 of the integers are less than 20.5, and 18 are greater, and so the greatest of them is 38.

2. The number of such sequences is the same as the number of ways a row of 10 dots can be separated by dividers. There are two choices for each of the 9 spaces between consecutive dots: either place a divider there or do not, so there are 2^9 possible placements of dividers. But this counting method includes the case where no dividers are placed, so there are $2^9 - 1$ = 511of the requested sequences.

   Alternatively, consider sequences whose sum is 3, 4, 5, ..., and look for a pattern.

3. The given condition implies that $.3 \le \frac{3}{d} < .4$, that is, $\frac{3}{10} \le \frac{3}{d} < \frac{4}{10}$. Thus $\frac{10}{3} \ge \frac{d}{3} > \frac{10}{4}$, so $10 \ge d > 7.5$. Therefore the possible values of d are 8, 9 and 10, and their sum is 27.

4. The quantity $x^2 + y^2$ is the square of the distance from the origin to the point with coordinates (x, y). The requested distance is minimized when that point is where a line through the origin is perpendicular to the line whose equation is given. Thus the minimum distance is the altitude to the hypotenuse of a right triangle whose legs have lengths 3 and 4. Let h be the length of the altitude, and let K be the area. Because the hypotenuse has length 5, $(1/2)\cdot 3\cdot 4 = K = (1/2)\cdot 5\cdot h$, so that $h = 12/5$, and $h2=14425$. So, p + q = 169.

5. Let O be the center of the smaller circle, and P the center of the larger. Then AB is twice the length of the altitude from A in triangle AOP. Note that OA = 10 and PA = 17, and use Heron's formula to find the area of ΔAOP is 84. Therefore the altitude from A has length 8, and AB = 16.
   Alternatively, one may notice that the length of the altitude from A is a leg of right triangles with hypotenuses 17 and 10, suggesting 8, 15, 17 and 6, 8, 10 triples. These work to yield an altitude with length 8 and AB = 16.

6. Let $x = \cos 20^\circ \cos 40^\circ \cos 80^\circ$. Then

$$\begin{aligned}
x\sin 20^\circ &= \sin 20^\circ \cos 20^\circ \cos 40^\circ \cos 80^\circ \\
&= (1/2)2\sin 20^\circ \cos 20^\circ \cos 40^\circ \cos 80^\circ \\
&= (1/2)\sin 40^\circ \cos 40^\circ \cos 80^\circ \\
&= (1/2)(1/2)2\sin 40^\circ \cos 40^\circ \cos 80^\circ \\
&= (1/2)(1/2)\sin 80^\circ \cos 80^\circ \\
&= (1/2)(1/2)(1/2)\sin 160^\circ = (1/8)\sin 20^\circ.
\end{aligned}$$

Thus $x = 1/8$.