Nassau County Interscholastic Mathematics League

Contest #4 Answers must be in simplest exact form, unless otherwise noted. 2010 – 2011

Calculators are allowed.

Time: 10 minutes

1. Find the closest integer to $\dfrac{2^{100}+3^{100}}{3^{100}}$.

2. Square *ABCD* is on the coordinate plane, *A* has coordinates (20, 11), and *B* has coordinates (75, -17). There are two possible positions for point *D*. Find the sum of the four coordinates of those two positions.

Time: 10 minutes

3. The greatest real number x such that $(12x-7)^2-3(12x-7)=18$, can be expressed in the form p/q, where p and q are relatively prime positive integers. Find $p+q$.

4. The length of each side of a right triangle is an integer. What is the sum of all possible lengths of the hypotenuse if the length of one of the legs is 12?

Time: 10 minutes

5. Find the maximum number of points of intersection of 5 different circles.

6. Find n be the smallest positive perfect square that is $1\frac{2}{3}$ times a perfect cube. Find the remainder when n is divided by 1000.

Solutions for Contest #4

1. The given expression is equivalent to $\left(\frac{2}{3}\right)^{100}+1$. Because $\left(\frac{2}{3}\right)^{100}$ is extremely close to 0, the closest integer to the given expression is 1.

2. Because A is the midpoint of the segment joining the two possible positions for point D, the sum of the four coordinates must be twice the sum of the coordinates of A, namely 2(20 + 11) = 62.

3. Let $y = 12x – 7$. Then $y^2 - 3y = 18$. Solve to obtain $y = 6$ or $y = -3$, then substitute to get $12x – 7 = 6$ or $12x – 7 = -3$. Then $x = 13/12$ or $x = 1/3$. The greater of these two numbers is 13/12, so $p + q = 25$.

4. Let the lengths of the legs be 12 and a and the length of the hypotenuse be c. So, *⍰2+122=⍰2*. So, *⍰+⍰ ⍰−⍰ =144*. Because a and c are integers with c > a, the only possible values of a and c are solutions of the following pairs of equations: *⍰+⍰=144, ⍰−⍰=1* ; *⍰+⍰=72, ⍰−⍰=2* ; *⍰+⍰=144, ⍰−⍰=1* ; *⍰+⍰=48, ⍰−⍰=3* ; *⍰+⍰=36, ⍰−⍰=4* ; *⍰+⍰=24, ⍰−⍰=6* ; *⍰+⍰=18, ⍰−⍰ =8* ; *⍰+⍰=16, ⍰−⍰=9* ; *⍰+⍰=12, ⍰−⍰=12* . Only the second, fourth, fifth, and sixth pair have solutions. So, the only possible values for c are 37, 20, 15, and 13. Their sum is 85.

5. Imagine the circles are drawn one at a time. The second circle intersects the first in at most two points. The third circle intersects each of the first two in at most two points, adding four more intersection points. Similarly, the fourth circle will add six more intersection points, and the fifth circle will add eight for a total of 2 + 4 + 6 + 8 =20 points.

6. Let a^2 be the requested perfect square, where a is a positive integer. Then for some positive integer b, $a^2 = 1\frac{2}{3}b^3$, that is, $3a^2 = 5b^3$. Let $m = 3a^2 = 5b^3$. Then m has 3 and 5 as prime factors. To find the minimum value of m, let $m = 3^x 5^y$, where x and y are positive integers. Because $a^2 = \frac{m}{3} = 3^{x-1}\cdot 5^y$, conclude that x is odd and y is even. And because $b^3 = \frac{m}{5} = 3^x \cdot 5^{y-1}$, conclude that x is a multiple of 3 and y is one more than a multiple of 3. The smallest such values of x and y are 3 and 4, respectively, so the smallest value for m is $3^3 \cdot 5^4$. Thus $a^2 = \frac{3^3\cdot 5^4}{3} = 3^2\cdot 5^4 = 5625$, and the requested remainder is 625.

Challenge: Find a general form for all such squares.