Nassau County Interscholastic Mathematics League

Contest #2 Answers must be integers from 0 to 999 inclusive. 2010 – 2011

Calculators are allowed.

Time: 10 minutes

1. Find the product of the real solutions of $x^{2010} = 1$.

2. In a cube whose edges have length $10\sqrt{6}$, *ABCD* and *EFGH* are opposite faces. Find the distance from *A* to the center of *EFGH*.

Time: 10 minutes

3. Find the least positive integer n such that 1040n is a square.

4. The lengths of the medians of a triangle are 9, 9, and 12. The area of the triangle can be expressed as $p\sqrt{q}$, where *p* and *q* are positive integers and q is not divisible by the square of a prime. Find *p* + *q*.

Time: 10 minutes

5. The coordinates of point *A* are (12, 77) and the coordinates of point *B* are (68, -53). Point *P* is on $\overline{AB}$ so that *AP*:*PB* = 3:7. The coordinates of *P* are (*m*, *n*). Find 10(*m* + *n*).

6. Find the coefficient of x^5 in the expansion of $(1 + x + x^2)^5$.

Solutions for Contest #2

1. The two real solutions are 1 and –1, and their product is –1.

2. Let O be the center of $EFGH$. Without loss of generality, let $ABFE$ be a face of the cube, and let P be the midpoint of $\overline{EF}$. Draw $\overline{OP}$ and $\overline{PA}$. Then
$PA^2 = PE^2 + EA^2 = \left(10\sqrt{6}\right)^2 + \left(5\sqrt{6}\right)^2 = 750$, and so
$AO^2 = AP^2 + PO^2 = 750 + \left(5\sqrt{6}\right)^2 = 900$. Thus $AO = 30$.

3. Notice that $1040 = 2^4 \cdot 5 \cdot 13$. The least possible value of n is therefore $5 \cdot 13 = 65$.

4. Label the triangle ABC, let K be its area, let $\overline{AD}$ and $\overline{BE}$ be the medians of length 9, and let $\overline{CF}$ be the median (and altitude) of length 12. Let G be the centroid of the triangle (the point of intersection of the medians). Because the medians of a triangle divide each other in the ratio 2:1, AG = 6 and GF = 4. Use the Pythagorean Theorem to conclude that $AF = 2\sqrt{5}$. Then the area of right triangle AFG is $(1/2)2\sqrt{5} \cdot 4 = 4\sqrt{5}$, and so $K = 6 \cdot 4\sqrt{5} = 24\sqrt{5}$. Thus $p + q =$ 29.

5. Each of the coordinates of P is the weighted average of the corresponding coordinates of A and B. In particular, the coordinates of P are
$\left(\frac{7 \cdot 12 + 3 \cdot 68}{10}, \frac{7 \cdot 77 + 3 \cdot -53}{10}\right) = (28.8, 38)$. Thus $10(m + n)$ =668.

6. The given product contains five factors of $\left(x^2 + x + 1\right)$. To expand the product, you must choose one term from 1, x and x^2 in each of the five factors. If you choose a 1's, b x's and c x^2's, then $a + b + c = 5$. In order for the product of the terms to be x^5, you must have $1^a \cdot x^b \cdot \left(x^2\right)^c = x^5$, that is, $b + 2c = 5$. Thus $a = c$, and so you must choose either no 1's, five x's and no x^2's; one 1, three x's and one x^2; or two 1's, one x and two x^2's.

   Count the three cases separately. In the first case, there is one way to choose no 1's, five x's and no x^2's. In the second case, there are 5 ways to choose a 1, then 4 ways to choose an x^2 for a total of 20 ways to choose one 1, three x's and one x^2. In the third case, there are 5 ways to choose one x, then $\binom{4}{2} = 6$ ways to choose two 1's for a total of 30 ways to choose two 1's, one x and two x^2's. Thus the coefficient of x^5 is 1 + 20 + 30 = 51.

Alternatively, we can count as follows:

Case I: choosing one x from each of the five trinomial factors, only one way
Case II: Choosing three x’s, one ⍰*2*, and one 1 is analogous to counting the number of arrangements of the letters in the word GEESE. 5!/(1!3!1!)
Case III: Choosing two ⍰*2* , one x and two 1’s is analogous to counting the number of arrangements of the letters in the word MAMAS. 5!/(2!1!2!)