Nassau County Interscholastic Mathematics League

Contest #3 Answers must be in simplest exact form, unless otherwise noted. 2009-2010

No calculators are allowed.

Time: 10 minutes

13. Compute the smallest positive integer, x, for which $5850x = N^3$, where N is an integer.
14. Solve for x: $3^x - 3^{x-2} = 648\sqrt{3}$.

Time: 10 minutes

15. Solve for x: $\log_5 x + \log_{25} x + \log_{125} x = \frac{22}{3}$.

16. Two adjacent sides of a parallelogram measure 7cm and 9cm, respectively. If one diagonal measures 11 cm, compute the length of the other diagonal.

Time: 10 minutes

17. If $\sin 2\theta = \frac{\sqrt{2}}{3}$, and θ is acute, compute the value of $(\sin\theta - \cos\theta)^4$.
18. If $9x + 40y = 360$, compute the minimum value of $\sqrt{x^2 + y^2}$.

Solutions for Contest #3

13) The prime factorization of 5850 is $5^2 \cdot 3^2 \cdot 13 \cdot 2$. The smallest factor needed to make $5850x$ a perfect cube is $x = 5 \cdot 3 \cdot 13^2 \cdot 2^2 = 10140$

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14) $3^x - 3^{x-2} = 648\sqrt{3};\ 3^{x-2}(9-1) = 648\sqrt{3};\ 3^{x-2} = 81\sqrt{3} \Rightarrow x - 2 = 4.5 \qquad x = 6.5$

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15) Use the base change law: $\frac{\log x}{\log 5} + \frac{\log x}{\log 25} + \frac{\log x}{\log 125} = \frac{22}{3}$. Then $\left(\frac{\log x}{\log 5}\right)\left(1 + \frac{1}{2} + \frac{1}{3}\right) = \frac{22}{3}$. Then $\left(\frac{\log x}{\log 5}\right)\frac{11}{6} = \frac{22}{3}$. Then $\log x = 4\log 5$ and x = 625.

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16) Call the angle opposite the diagonal θ. Using the Law of Cosines: $121 = 81 + 49 - 126\cos\theta \rightarrow -9 = -126\cos\theta \rightarrow$ $\cos\theta = \frac{1}{14}$ The angle opposite the other diagonal is supplementary to θ. Its cosine is $-\frac{1}{14}$. Use the Law of Cosines again with the length of the other diagonal = x.

$$x^2 = 81 + 49 - 126\left(-\frac{1}{14}\right) = 139;\ \ x = \sqrt{139}$$

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17. $$(\sin\theta - \cos\theta)^2 = \sin^2\theta - 2\sin\theta\cos\theta + \cos^2\theta = 1 - \frac{\sqrt{2}}{3}$$

$$(\sin\theta - \cos\theta)^4 = \left(1 - \frac{\sqrt{2}}{3}\right)^2 = 1 - \frac{2\sqrt{2}}{3} + \frac{2}{9} = \frac{11}{9} - \frac{2\sqrt{2}}{3} = \frac{11 - 6\sqrt{2}}{9}$$

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18) $\sqrt{x^2 + y^2}$ represents the distance from the origin to a point on the line $9x + 40y = 360$. The shortest distance to line from the origin lies along the perpendicular to the line, or the altitude to the hypotenuse of the right triangle in the first quadrant by the line and the coordinate axes. The area of this triangle is 180. Its hypotenuse is 41. The to the hypotenuse is $\frac{360}{41}$.